PERGAMON

International Journal of Solids and Structures 38 (2001) 9493–9503

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Application of transversely isotropic materials to multi-layer shell elements undergoing finite rotations and large strains

Daniela Lürding *, Yavuz Başar, Ulrike Hanskötter

Institute for Structural Mechanics, Ruhr-University Bochum, 1A 6/135, D-44780 Bochum, Germany

Received 4 November 1999; in revised form 4 November 1999

Abstract

The contribution deals with an extension of a classical Neo–Hookean model for compressible isotropic materials to transverse isotropy. With this enhancement for one preferred material direction there is a possibility to simulate large strains in volume changes of the isotropic basic continuum and supplementary in fiber direction. The integrity basis of polynomial invariants in case of transversely isotropic hyperelasticity consists of three principal invariants of the isotropic basic continuum and additionally of two principal invariants for the preferred material direction. The proposed stored energy function for transverse isotropy contains the classical theory near to the natural state and fulfills the restriction on polyconvexity and coerciveness.

By numerical enforcement of the material model into shell kinematics without rotational variables a four-node isoparametric finite element is developed using special concepts to avoid locking. The capability of the algorithms proposed is demonstrated by a numerical example involving large strains as well as finite rotations. © 2001 Published by Elsevier Science Ltd.

Keywords: Isotropic; Shell; Strains; Kinematics

1. Introduction

The main advantage of anisotropic materials is their optimized design regarding the functionality of the structure (Braun, 1995; Schultz, 1996; Schröder, 1996; Başar et al., 1999a,b). Dealing with large strains possible fields of application are automobile tires and biological soft tissues like arteries, skin or muscles. Especially the aorta is a multi-layered structure composed of very thin transversely isotropic soft tissue layers of muscles. In case of internal blood pressure the arteries undergo finite rotations as well as large strains characterized by considerable changes of initial side length coupled with a considerable change of thickness.

Analyzing finite deformations a special attention will be given to model isotropy and transverse isotropy in case of hyperelasticity. The use of a Lagrangean formulation in the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$

* Corresponding author.

E-mail addresses: daniela.luerding@consult.hochtief.de, d.luerding@sd.ruhr-uni-bochum.de (D. Lürding).

0020-7683/01/$ - see front matter © 2001 Published by Elsevier Science Ltd.
PII: S0020-7683(01)00134-2

performs the principle of frame indifference. Symmetries of the material restrict the formulation to an irreducible integrity basis of polynomial invariants, which consists for transverse isotropy of five irreducible invariants. In literature basic invariants are often used in this context (Boehler, 1987; Spencer, 1971; Xiao, 1996). With the geometrical meaning of principal invariants, the presented formulation contains three principal invariants corresponding to the isotropic basic continuum and two principal invariants regarding the preferred material direction. They can be associated with longitudinal changes of length in fiber direction and with changes of the orientation in a 3d-continuum. In case of infinitesimal strains near to the natural state any material model must be reducible to the classical theory. Furthermore, simulating large strains in hyperelasticity (Ogden, 1972; Başar and Ding, 1997; Wriggers and Reese, 1996; Betsch et al., 1996), the restrictions of *polyconvexity* and *coerciveness* (Ball, 1977; Ciarlet, 1988) should be taken into consideration. One possible stored energy function for transversely isotropic, compressible materials is developed fulfilling the restrictions on constitutive relations.

The application of a continuum based multi-layer finite shell element provides the simulation of varying transversely isotropic material components throughout the thickness very accurately. Thereby *assumed strain concepts* are used to eliminate *shear*- and *curvature-locking* and *enhanced assumed strain concepts* to avoid Poisson-, *membrane*- and *volume-locking* (Betsch and Stein, 1995; Bischoff and Ramm, 1997; Eckstein, 1999).

2. Analysis of nonlinear deformation

The following relations will be presented in tensor notation. Latin indices represent the number $\{1,2,3\}$ and Greek ones the number $\{1,2\}$ in combination with standard summation convention. A Lagrangean point X is identified with its position vector $\mathbf{X} = \mathbf{X}(\Theta^1, \Theta^2, \Theta^3)$ depending on the curvilinear coordinates Θ^i in a three-dimensional Euclidean space $\mathbb{R}^3$. Whereas the undeformed state is characterized with capital letters, the deformed configuration is written in small ones. The covariant basic vectors

$$\mathbf{G}_i = \frac{\partial \mathbf{X}}{\partial \Theta^i} \quad \text{and} \quad \mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \Theta^i} \tag{1}$$

are coupled with the contravariant basic vectors $\mathbf{G}^i = G^{ij}\mathbf{G}_j$, $\mathbf{g}^i = g^{ij}\mathbf{g}_j$, $\delta^i_j = \mathbf{G}_j \cdot \mathbf{G}^i = \mathbf{g}_j \cdot \mathbf{g}^i$ and the metric is given by $G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j$, $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ as well as $G^{ij} = (G_{ij})^{-1}, g^{ij} = (g^{ij})^{-1}$. A deformation of the reference configuration $\bar{\Omega}$ is a vector field $\boldsymbol{\varphi} : \bar{\Omega} \rightarrow \mathbb{R}^3$ that is *smooth enough, injective except possibly on the boundary of the set* $\bar{\Omega}$, and *orientation-preserving* with a deformation gradient $\mathbf{F} = \nabla\boldsymbol{\varphi} \in \mathbb{M}^3_+$, $\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i$ performing the condition $\det \mathbf{F} > 0$. The computation of a differential *length, surface* and *volume* element of the deformed configuration in terms of same quantities defined in the reference configuration are necessary for the nonlinear strain–stress relation based on a Lagrangean description.

$$\begin{aligned}
&\text{Length (1d)} \quad \mathrm{d}s = (\mathrm{d}\mathbf{X}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\,\mathrm{d}\mathbf{X})^{1/2},\\
&\text{Surface (2d)} \quad \mathrm{d}a = \|\mathrm{Cof}\,\mathbf{F}\mathbf{N}\|\mathrm{d}A, \quad \mathbf{N} \text{ normal to } \mathrm{d}A,\\
&\text{Volume (3d)} \quad \mathrm{d}v = \det\mathbf{F}\,\mathrm{d}V.
\end{aligned} \tag{2}$$

The geometrical meaning of the principal invariants of the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is remarkable. The first principal invariant $I_{\mathbf{C}} = \|\mathbf{F}\|^2$ can be associated with the longitudinal length changes of differential line elements, the second one $II_{\mathbf{C}} = \|\mathrm{Cof}\,\mathbf{F}\|^2$ is related to a change of the angle between two differential line elements caused by shear deformations and the third principal invariant $III_{\mathbf{C}} = (\det\mathbf{F})^2$ describes volume changes of a 3d-continuum. Normally, the principal invariants are calculated in terms of the basic invariants $\mathrm{tr}\,\mathbf{C}$, $\mathrm{tr}\,\mathbf{C}^2$, $\mathrm{tr}\,\mathbf{C}^3$ taking the Cayley–Hamilton theorem into consideration:

$$
\begin{aligned}
I_{\mathbf{C}} &= \operatorname{tr}\mathbf{C},\\
II_{\mathbf{C}} &= \operatorname{tr}\operatorname{Cof}\mathbf{C} = \tfrac{1}{2}[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2], \quad \operatorname{Cof}\mathbf{C} = (\det\mathbf{C})\ \mathbf{C}^{-1},\\
III_{\mathbf{C}} &= \det\mathbf{C} = \tfrac{1}{3}\operatorname{tr}\mathbf{C}^3 - \tfrac{1}{2}(\operatorname{tr}\mathbf{C}^2)\operatorname{tr}\mathbf{C} + \tfrac{1}{6}(\operatorname{tr}\mathbf{C}^3).
\end{aligned}
\tag{3}
$$

3. Modeling of hyperelastic material

An elastic material is hyperelastic if a stored energy function $\hat{W} : \bar{\Omega} \times \mathbb{M}^3_+ \to \mathbb{R}$ exists. The material has to be homogeneous and the reference configuration has to be the natural state. Therefore one demand for general constitutive relations is the frame-indifference

$$\hat{W}(\mathbf{F}) = \hat{W}(\mathbf{QF}) = \tilde{W}(\mathbf{F}^{\mathrm{T}}\mathbf{F}) = \tilde{W}(\mathbf{C}) \quad \text{for all } \mathbf{Q} \in \mathbb{O}^3_+, \tag{4}$$

which implies an explicit dependence on the right Cauchy–Green tensor.

The demand for frame indifference works on the deformed configuration, material symmetry is regarded to the undeformed state. The group of symmetric transformations $\mathbb{G}_X$ characterizes the symmetry properties of the material in the undeformed state and imposes restrictions upon the manner, in which $\tilde{W}$ depends on $\mathbf{C}$ for constructing an integrity basis of irreducible polynomial invariants (Green and Adkins, 1960; Spencer, 1971)

$$\tilde{W}(\mathbf{C}) = \tilde{W}(\mathbf{QCQ}^{\mathrm{T}}) \quad \text{for all } \mathbf{Q} \in \mathbb{G}_X \subseteq \mathbb{O}^3. \tag{5}$$

The 2. Piola-Kirchhoff stresses and the elasticity tensor

$$\tilde{\mathbf{S}} = 2\frac{\partial \tilde{W}}{\partial C_{ij}}\mathbf{A}_i \otimes \mathbf{A}_j \quad \text{and} \quad \tilde{\mathbb{C}} = 4\frac{\partial^2 \tilde{W}}{\partial C_{ij}\partial C_{kl}}\mathbf{A}_i \otimes \mathbf{A}_j \otimes \mathbf{A}_k \otimes \mathbf{A}_l \tag{6}$$

are calculated corresponding to a local orthogonal, right-handed coordinate system (Green and Adkins, 1960; Schultz, 1996)

$$
\begin{aligned}
\mathbf{A}_1 &= \frac{\sin(\vartheta - \alpha)}{\sin\vartheta}\mathbf{G}_{\langle 1\rangle} + \frac{\sin\alpha}{\sin\vartheta}\mathbf{G}_{\langle 2\rangle},\\
\mathbf{A}_2 &= -\frac{\cos(\vartheta - \alpha)}{\sin\vartheta}\mathbf{G}_{\langle 1\rangle} + \frac{\cos\alpha}{\sin\vartheta}\mathbf{G}_{\langle 2\rangle}
\end{aligned}
\tag{7}
$$

with $\mathbf{A}_1$ in preferred material direction, $\mathbf{G}_{\langle i\rangle} = \mathbf{G}_i/\sqrt{G_{ii}}$, $\|\mathbf{A}_i\| = 1$, $\mathbf{A}_3 = \mathbf{A}_1 \times \mathbf{A}_2$, $\cos\vartheta = \mathbf{G}_{\langle 1\rangle} \cdot \mathbf{G}_{\langle 2\rangle}$ and the fiber direction α with $\cos\alpha = \mathbf{G}_{\langle 1\rangle} \cdot \mathbf{A}_1$. Related to such a system of basic vectors $\mathbf{A}_i$ the elasticity tensor contains a minimal number of non-zero components with the symmetries

$$\tilde{C}^{ijkl} = \tilde{C}^{jikl} = \tilde{C}^{klij} = \tilde{C}^{ijlk} \tag{8}$$

and 21 independent components at the most. The smaller the symmetries the material include, the bigger is the number of independent components of $\tilde{C}^{ijkl}$, and just so is the integrity basis of irreducible invariants as well as the number of material coefficients.

3.1. Isotropic hyperelasticity

If the group of symmetric transformations contains the whole orthogonal group $\mathbb{G}_X = \mathbb{O}^3$, a stored energy function can be presented in terms of the eigenvalues $\lambda_i(\mathbf{C})$ or of their elementary symmetric functions $\{I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}\}$ (Başar and Weichert, 2000)

$$\tilde{W}(\mathbf{C}) = \tilde{W}(\mathbf{QCQ}^{\mathrm{T}}) = \acute{W}(\lambda_1, \lambda_2, \lambda_3) = \breve{W}(I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}). \tag{9}$$

3.2. Transverse isotropic hyperelasticity

With one preferred material direction the symmetric transformation group $\mathbb{G}_X$ contains all continuous rotations related to the fiber direction and reflections with respect to the in plane fiber (Green and Adkins, 1960; Spencer, 1971). On the base of a known set of symmetric transformations $\mathbb{G}_X$ an integrity basis of polynomial, irreducible invariants can be specified, which contains five polynomial invariants in the case of transverse isotropy

$$\tilde{W}(\mathbf{C}) = \tilde{W}(\mathbf{QCQ}^{\mathrm{T}}) = \breve{W}(I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}, C_{11}, C_{1K}C_{K1}) \quad \mathbf{Q} \in \mathbb{G}_X \tag{10}$$

with $K = 2, 3$. An isotropic stored energy function can be constructed with a structural tensor $\mathbf{A}_1 \otimes \mathbf{A}_1$ and $\|\mathbf{A}_1\| = 1$ referring to the preferred material direction.

$$\begin{aligned} \tilde{W}(\mathbf{C}, \mathbf{A}_1 \otimes \mathbf{A}_1) &= \tilde{W}(\mathbf{QCQ}^{\mathrm{T}}, \mathbf{QA}_1 \otimes \mathbf{A}_1\mathbf{Q}^{\mathrm{T}}) \quad \mathbf{Q} \in \mathbb{O}^3 \\ &= \breve{W}(I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}, I_{\mathbf{A}}, II_{\mathbf{A}}). \end{aligned} \tag{11}$$

The three principal invariants $\{I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}\}$ are known from an isotropic continuum and the additional two ones

$$\begin{aligned} I_{\mathbf{A}} &= \mathrm{tr}\,[\mathbf{C}(\mathbf{A}_1 \otimes \mathbf{A}_1)] = \mathbf{A}_1\mathbf{C}\mathbf{A}_1 = C_{11}, \\ II_{\mathbf{A}} &= \mathrm{tr}\,\mathrm{Cof}[\mathbf{C}(\mathbf{A}_1 \otimes \mathbf{A}_1)] = \tfrac{1}{2}[(\mathbf{A}_1\mathbf{C}\mathbf{A}_1)^2 - \mathbf{A}_1\mathbf{C}^2\mathbf{A}_1] = C_{1K}C_{K1} \end{aligned} \tag{12}$$

can be associated with the deformation of a differential line element in fiber direction with respect to its length and its orientation in a 3d-continuum. The formulation includes the assumption that even the deformed state contains only one preferred material direction.

4. Strain energy function for large strains

It is desirable that constitutive equations reflect in some fashion the intuitive idea that "*infinite stresses must accompany extreme strains*" (Antman, 1983a,b). This corresponds in the case of hyperelasticity to the requirement (Ciarlet, 1988) that the stored energy function $\hat{W}$ approaches $+\infty$ as

$$\begin{aligned} &\lambda_i(\mathbf{C}) \to 0^+ \iff \det \mathbf{F} \to 0^+, \\ &\lambda_i(\mathbf{C}) \to +\infty \iff \|\mathbf{F}\| \to +\infty \wedge \|\mathrm{Cof}\,\mathbf{F}\| \to +\infty \wedge \det \mathbf{F} \to +\infty. \end{aligned} \tag{13}$$

A stored energy function $\hat{W}$ is *polyconvex* (Ciarlet, 1988; Ball, 1977) if there exists a convex function $\mathbb{W} : \mathbb{M}^3 \times \mathbb{M}^3 \times]0, +\infty[\to \mathbb{R}$ such that

$$\hat{W}(\mathbf{F}) = \mathbb{W}(\mathbf{F}, \mathrm{Cof}\,\mathbf{F}, \det \mathbf{F}) \quad \text{for all } \mathbf{F} \in \mathbb{M}^3_+. \tag{14}$$

With the behavior as $\lim_{(\det \mathbf{F}) \to 0^+} W(\mathbf{F}) = +\infty$ and the performance of the *coerciveness inequality*

$$\hat{W}(\mathbf{F}) \geqslant \alpha\{\|\mathbf{F}\|^p + \|\mathrm{Cof}\,\mathbf{F}\|^q + (\det \mathbf{F})^r\} + \beta \quad \alpha > 0, \ \ \beta \in \mathbb{R}, \tag{15}$$

which bases on proofs of existence (Ball, 1977) as well as the sharper restrictions for the exponents $p \geqslant 2$, $q \geqslant p/(p-1)$, $r > 0$, there exists a deformation $\boldsymbol{\varphi}$ with $\nabla\boldsymbol{\varphi} = \mathbf{F}$, which solve the minimization problem for a large strain analysis. The physical and mathematical restrictions must be fulfilled for isotropy and for transverse isotropy.

4.1. A stored energy function for isotropy

Combining these aspects, the stored energy function of Mooney–Rivlin type for compressible materials

$$\tilde{W}(\mathbf{C}) = c_1(I_{\mathbf{C}} - 3) + c_2(II_{\mathbf{C}} - 3) + \Gamma(III_{\mathbf{C}}) \quad c_1, c_2 > 0 \tag{16}$$

is *polyconvex* and satisfies the corresponding *coerciveness inequality* with $p = q = 2$. The exponent r depends on the convex function $\Gamma :]0, +\infty[\to \mathbb{R}$, which is able to simulate the behavior as $\lim_{(\det \mathbf{F}) \to 0^+} W(\mathbf{F}) = +\infty$. A special case is the Neo–Hookean model for compressible materials

$$\hat{W}(\mathbf{F}) = \tilde{W}(\mathbf{C}) = c_1(I_{\mathbf{C}} - 3) + \Gamma(III_{\mathbf{C}}), \tag{17}$$

which will be used to extend the stored energy function to transverse isotropy.

4.2. Large strains in a preferred material direction

With the existence of one preferred material direction inside the isotropic basic continuum two additional limiting states of strain must be taken into account, namely extreme stretching and extreme compression of the fiber described by

$$\hat{W}(\mathbf{F}, \mathbf{A}_1 \otimes \mathbf{A}_1) \to +\infty \quad \text{for } I_{\mathbf{A}} \to +\infty \text{ and } I_{\mathbf{A}} \to 0^+. \tag{18}$$

Using principal invariants a stored energy function for transverse isotropy fulfills the restriction on polyconvexity (14) and coerciveness (15). The second additional invariant $II_{\mathbf{A}}$ (12) permits the simulation of deformations regarding the orientation of the fiber in a 3d-continuum. This part is especially necessary for the reduction of an arbitrary constitutive equation to the classical theory near to the natural state.

4.3. A stored energy function for transverse isotropy

One possible stored energy function being able to simulate large elastic strains in the isotropic basic continuum and additionally in the preferred material direction can be constructed as follows:

$$\tilde{W}(\mathbf{C}, \mathbf{A}_1 \otimes \mathbf{A}_1) = \tfrac{1}{2}\mu_\perp(I_{\mathbf{C}} - 3) + \Gamma(III_{\mathbf{C}}) + \tfrac{1}{4}\alpha(I_{\mathbf{C}} - 3)(I_{\mathbf{A}} - 1) - (\mu_\parallel - \mu_\perp)II_{\mathbf{A}} + \Gamma^*(I_{\mathbf{A}}). \tag{19}$$

Volume changes and stretching of the fiber are described by same convex function with the *epigraph* for simulating stretching and the *natural logarithm* for simulating compression

$$\begin{aligned} \Gamma(III_{\mathbf{C}}) &= \tfrac{1}{2}\lambda_\perp[(III_{\mathbf{C}})^{1/2} - 1]^2 - \tfrac{1}{2}\mu_\perp \ln III_{\mathbf{C}}, \\ \Gamma^*(I_{\mathbf{A}}) &= \tfrac{1}{2}\lambda_\parallel[(I_{\mathbf{A}})^{1/2} - 1]^2 - \tfrac{1}{2}\mu_\parallel \ln I_{\mathbf{A}} \end{aligned} \tag{20}$$

with $0 < I_{\mathbf{A}}, III_{\mathbf{C}} < 4$. The terms $(I_{\mathbf{A}} - 1)(I_{\mathbf{C}} - 3)$ and $(-\mu_\perp \ln III_{\mathbf{C}} - \mu_\parallel \ln I_{\mathbf{A}}) = \ln(I_{\mathbf{A}}^{\mu_\parallel} \cdot III_{\mathbf{C}}^{\mu_\perp})$ couple volume changes with longitudinal changes in the preferred direction. The function depends on five independent material parameters, which are listed in Appendix A.

5. Constitutive relation near to the natural state

Near to the unstressed reference configuration the difference of the 2. Piola–Kirchhoff stresses $\tilde{\mathbf{S}}(\mathbf{I} + 2\mathbf{E}) - \tilde{\mathbf{S}}(\mathbf{I})$ in terms of the right Cauchy–Green tensor is a measure of the discrepancy between deformation and rigid deformation ($\mathbf{C} = \mathbf{I}$) (Ciarlet, 1988), which is analogous to the Green–Lagrange strains $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$.

For the natural state, which is characterized by $I_{\mathbf{C}} = II_{\mathbf{C}} = 3$, $III_{\mathbf{C}} = 1$, $I_{\mathbf{A}} = 1$, $II_{\mathbf{A}} = 0$, the stored energy must be zero

$$\tilde{W}(I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}, I_{\mathbf{A}}, II_{\mathbf{A}})\Big|_{I_{\mathbf{C}}=II_{\mathbf{C}}=3,\ III_{\mathbf{C}}=I_{\mathbf{A}}=1,\ II_{\mathbf{A}}=0} = 0. \tag{21}$$

The unstressed state must be verified

$$\mathbf{S} = 2\left[\frac{\partial}{\partial \mathbf{C}}\tilde{W}(I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}, I_{\mathbf{A}}, II_{\mathbf{A}})\right]\Bigg|_{I_{\mathbf{C}}=II_{\mathbf{C}}=3,\ III_{\mathbf{C}}=I_{\mathbf{A}}=1,\ II_{\mathbf{A}}=0} = 0. \tag{22}$$

Finally, any material formulation has to return into the classical theory (Spencer, 1984)

$$\mathbf{S} = (\lambda \operatorname{tr}\mathbf{E} + \alpha \mathbf{A}_1\mathbf{E}\mathbf{A}_1)\mathbf{I} + (\alpha \operatorname{tr}\mathbf{E} + \beta \mathbf{A}_1\mathbf{E}\mathbf{A}_1)\mathbf{A}_1 \otimes \mathbf{A}_1 + 2\mu_\perp \mathbf{E} + 2(\mu_\parallel - \mu_\perp)(\mathbf{A}_1 \otimes \mathbf{A}_1\mathbf{E} + \mathbf{E}\mathbf{A}_1 \otimes \mathbf{A}_1), \tag{23}$$

which is evaluated for infinitesimal strains near to the natural state.

6. Finite element formulation

6.1. Shell kinematics

The basic assumption of the presented continuum based shell kinematics is the approximation of the position vector $\mathbf{x} = \mathbf{x}(\Theta^i)$ by a linear expansion in thickness coordinates Θ^3

$$\mathbf{x} = \overset{0}{\mathbf{x}} + \Theta^3\mathbf{d} \quad -h/2 \leqslant \Theta^3 \leqslant +h/2 \tag{24}$$

with the position vector of the midsurface $\overset{0}{\mathbf{x}} = \overset{0}{\mathbf{x}}(\Theta^\alpha)$. The extensible shell director $\mathbf{d} = \mathbf{d}_0(\Theta^\alpha)$ enables to simulate thickness changes of the structure starting from $\|\mathbf{D}\| = 1$ in the undeformed state. The presented six-parameter single layer theory can be applied to a multi-layer shell concept requiring C^0-continuity of the displacement field on all interfaces (Başar et al., 1993, 1999a,b).

6.2. Virtual work

The total energy $\Pi = \Pi_i + \Pi_e$ depends on conservative loads in Π_e and the above mentioned stored energy function (19) for transverse isotropy in Π_i. During the linearization procedure of $\delta\Pi_i = \int_\Omega \delta\mathbf{E}{:}\mathbf{S}\,\mathrm{d}\Omega$ the following parts:

$$\Delta\delta\Pi_i = \int_\Omega \Delta\delta\mathbf{E} : \mathbf{S}\,\mathrm{d}\Omega + \int_\Omega \delta\mathbf{E} : \Delta\mathbf{S}\,\mathrm{d}\Omega + \int_\Omega \delta\mathbf{E} : \mathbf{S}\,\mathrm{d}\Omega \tag{25}$$

contain the geometrical and physical tangential element stiffness matrix in the first two terms and the vector of the internal forces in the last one. With the presented hyperelastic material model for compressible, transversely isotropic materials, the 2. Piola–Kirchhoff stresses are given by

$$\begin{aligned}\tilde{\mathbf{S}} = {} & [\mu_\perp + 0.5\alpha(I_{\mathbf{A}} - 1)]\mathbf{I} + [0.5\alpha(I_{\mathbf{C}} - 3) - \mu_\parallel(I_{\mathbf{A}})^{-1} + \lambda_\parallel(1 - (I_{\mathbf{A}})^{-1/2})]\mathbf{A}_1 \otimes \mathbf{A}_1 - 2(\mu_\parallel - \mu_\perp) \\ & \times (\mathbf{A}_1\mathbf{C} \otimes \mathbf{A}_1 + \mathbf{A}_1 \otimes \mathbf{C}\mathbf{A}_1) + [-\mu_\perp + \lambda_\perp(III_{\mathbf{C}} - (III_{\mathbf{C}})^{1/2})]\mathbf{C}^{-1}\end{aligned} \tag{26}$$

and the elasticity tensor by

$$\tilde{\mathbb{C}} = \alpha[\mathbf{I} \otimes (\mathbf{A}_1 \otimes \mathbf{A}_1) + (\mathbf{A}_1 \otimes \mathbf{A}_1) \otimes \mathbf{I}] + 2(\mu_{\|}(I_{\mathbf{A}})^{-2} + 0.5\lambda_{\|}(I_{\mathbf{A}})^{-3/2})\mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{A}_1$$
$$+ 2\lambda_{\perp}(III_{\mathbf{C}} - 0.5(III_{\mathbf{C}})^{1/2})\mathbf{C}^{-1} \otimes \mathbf{C}^{-1} + 2[\mu_{\perp} - \lambda_{\perp}(III_{\mathbf{C}} - (III_{\mathbf{C}})^{1/2})]\mathbb{I}_{\mathbf{C}^{-1}} - 4(\mu_{\perp} - \mu_{\|})\mathbb{A} \qquad (27)$$

with $\mathbb{A} = (A^i A^k G^{jl} + A^j A^l G^{ik})\mathbf{A}_i \otimes \mathbf{A}_j \otimes \mathbf{A}_k \otimes \mathbf{A}_l$.

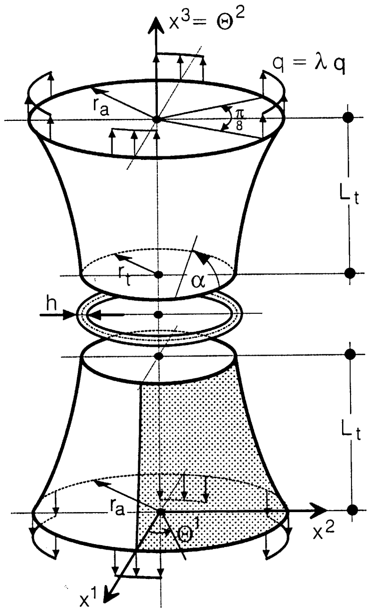
6.3. Discretization

The theoretical fundamentals are transformed into a four-node layer-wise isoparametric shell element using bilinear shape functions for interpolating all displacement variables. Locking phenomena derive mostly from insufficient approximation of the strains caused by a low-order approximation of the displacement field. *Assumed strain concepts* are used in *shear-locking* related to constant transverse shear components and in case of *curvature-locking*, related to transverse normal strains. *Enhanced assumed strain concepts* are applied to *membrane-* and *volume-locking* referred to the constant tangential strain components and additionally to avoid Poisson-*locking* by introducing linear transverse normal strains (Betsch, 1996; Wriggers and Reese, 1996; Bischoff and Ramm, 1997; Eckstein, 1999).

7. Examples

The hyperbolical shell (Fig. 1) with locally distributed vertical line loads renders the analysis of large strain phenomena as well as finite rotations. Due to the symmetry of the structure one eighth of the shell is discretized by 12 × 12 elements. The nonlinear simulation is done with the developed transversely isotropic model (19) and with a corresponding model based on the classical theory (23) (Spencer, 1984).

Starting with a nearly incompressible continuum the material properties for the preferred material direction are varied with respect to a factor Δ (Figs. 2 and 3). With fibers in Θ^1- or Θ^2-direction the different



Geometry:

$$r(\Theta^2) = \sqrt{r_t^2 + (\Theta^2 - L_t)^2 \, \frac{r_a^2 - r_t^2}{L_t^2}}$$

$r_a = 5.0$ $\quad r_t = 3.0$ $\quad L_t = 6.0$

$h = 0.5$ $\quad h_1 = 0.125$ $\quad h_2 = 0.25$

Line loads: $q = 1.0$

Material data:

$E_{\|} = 165.0$ $\quad E_{\perp} = 6.6$

$\mu_{\|} = 55.4$ $\quad \mu_{\perp} = 2.2$

$\nu_{\|} = 0.49$

Fig. 1. Hyperboloidal shell under concentrated loads.

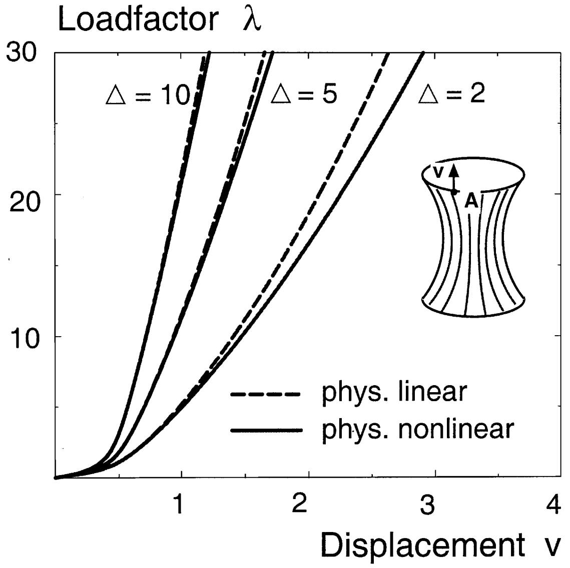


Fig. 2. Hyperboloidal shell: load displacement diagram with $E_{\|} = \Delta E$, $\mu_{\|} = \Delta\mu$.

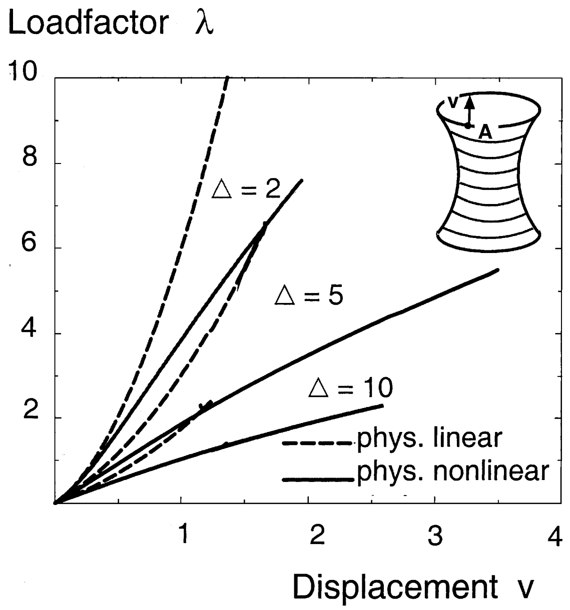


Fig. 3. Hyperboloidal shell: load displacement diagram with $E_{\perp} = E/\Delta$, $\mu_{\perp} = \mu/\Delta$.

behavior of a physical linear (23) and nonlinear model (19) is shown. If the load and the fiber are oriented in the same direction, it is obvious that the bigger the differences of matrix and fiber with increasing stiffness factor Δ are the inferior is the influence of the physical nonlinearities. But regarding fibers perpendicular to the load direction, it is evident that the bigger the differences of the matrix and fiber with increasing stiffness factor Δ are the bigger is the influence of the physical nonlinearities.

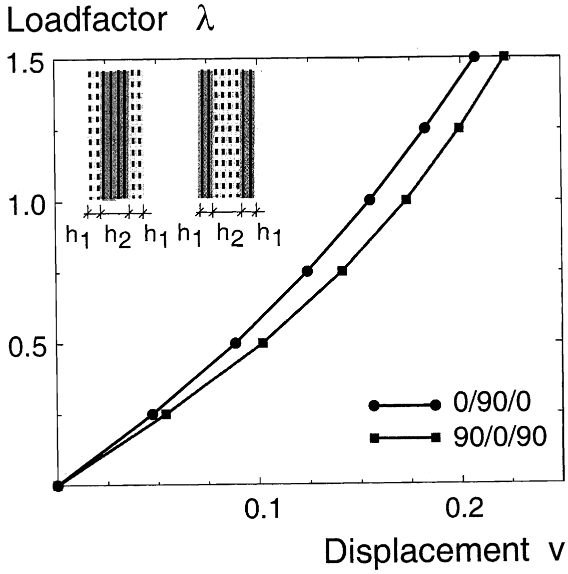


Fig. 4. Hyperboloidal shell: load displacement diagram with $E_{\|} = 5E$, $\mu_{\|} = 5\mu$ and $E_{\perp} = 0.2E$, $\mu_{\perp} = 0.2\mu$.

Another numerical simulation is carried out for two differently twisted stacking sequences of a three layered shell structure (Fig. 4), namely [0°/90°/0°] and [90°/0°/90°]. Differences in global deformation behavior cannot be seen. The local effects only appear if the mesh is refined towards the local concentrations (Başar et al., 1999a,b).

8. Conclusion

A general mathematical concept for considering transversely isotropic hyperelastic materials starting from a group of symmetric transformations has been presented. Five principal invariants, the three of the isotropic basic continuum and two regarding the preferred material direction, has been chosen for constructing the integrity basis of polynomial irreducible invariants taking their geometrical meaning into account. Moreover, the use of principal invariants has rendered the incorporation of the integrity basis for transverse isotropy into the concepts of polyconvexity and coerciveness. The developed stored energy function is valid for transversely isotropic materials, which are able to sustain large elastic strains. This formulation includes the classical theory in case of infinitesimal strains and contains a Neo–Hookean model for compressible isotropic materials as a special case. Furthermore, the developed material model is able to simulate large strains additionally in fiber direction. First application to a shell model has been pointed out.

Acknowledgements

The authors gratefully acknowledge the financial support of the German National Science Foundation within the SFB 398 and the Graduate Collegue *Computational Structural Dynamics*.

Appendix A

For transversely isotropic material behavior there are five independent material parameters, which are parallel and perpendicular to the preferred material direction with Young's moduli $E_{\|}$, $E_{\perp}$, Lame's shear moduli $\mu_{\|}$, $\mu_{\perp}$, Poisson's ratio $v_{\|}$. The Poisson's ratio $v_{\perp}$ can be calculated in the plane of isotropy $v_{\perp} = E_{\perp}/[2(1+\mu_{\perp})]$ (Jones, 1975). For the developed stored energy function $\mu_{\|}$ and $\mu_{\perp}$ are used as well as

$$\begin{aligned} \lambda_{\perp} &= \frac{1}{\Delta}(v_{\perp} + n v_{\|}^2)E_{\perp}, \\ \lambda_{\|} &= \frac{1}{4\Delta}((1 - v_{\perp}^2)E_{\|} - 2\mu_{\perp} - 4\mu_{\|} - 2(1+v_{\perp})v_{\|}E_{\perp} + (v_{\perp} + n v_{\|}^2)E_{\perp}), \\ \alpha &= \frac{1}{\Delta}((1+v_{\perp})v_{\|} - v_{\perp} - n v_{\|}^2)E_{\perp} \end{aligned} \tag{A.1}$$

with $\Delta = 1 - 2nv_{\|}^2 - v_{\perp}^2 - 2v_{\perp}v_{\|}^2$ and $n = E_{\perp}/E_{\|}$.

The parameter β in Eq. (23) is calculated by

$$\beta = \frac{1}{\Delta}(n(3v_{\|}^2 + 1 - v_{\perp}^2) + 2v_{\perp}(1 - v_{\|}) - 2v_{\|} - 1)E_{\perp} - 4\mu_{\|}. \tag{A.2}$$

References

Antman, S.S., 1983a. The influence of elasticity on analysis: Modern developments. Bull. Amer. Mach. Soc. 9, 267–291.

Antman, S.S., 1983b. Regular and singular problems for large elastic deformations of tubes, wedges, and cylinders. Arch. Rational Mech. Anal. 83, 1–52.

Başar, Y., Weichert, D., 2000. Nonlinear Continuum Mechanics of Solids. Springer, New York.

Başar, Y., Ding, Y., 1997. Shear deformation models for large strain shell analysis. Int. J. Solids Struct. 34 (14), 1687–1708.

Başar, Y., Ding, Y., Schultz, R., 1993. Refined shear-deformation models for composite laminates with finite rotations. Int. J. Sol. Struct. 30 (19), 2611–2638.

Başar, Y., Hanskötter, U., Lürding, D., 1999. Application of adaptive mesh refinement to composite shell structures. European Conference on Computational Mechanics ECCM'99. München.

Başar, Y., Itzkov, M., Eckstein, A., 1999. Composite laminates: Nonlinear interlaminar stress analysis by multi-layer shell elements. Comp. Meth. Engng. Aspects in Modern Computational Structural Analysis. invited paper.

Ball, J.M., 1977. Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. 63, 337–403.

Betsch, P., 1996. Statische und dynamische Berechnung von Schalen endlicher elastischer Deformationen mit gemischten finiten Elementen. Forschungs- und Seminarberichte aus dem Bereich der Mechanik, Mitteilung Nr.96-4, Universität Hannover.

Betsch, P., Gruttmann, F., Stein, E., 1996. A 4-node finite shell element for the implementation of general hyperelastic 3D-elasticity at finite strains. Comp. Meth. Appl. Mech. Engng. 130, 57–79.

Betsch, P., Stein, E., 1995. An assumed strain approach avoiding artificial thickness straining for a non-linear 4-node shell element. Comm. Num. Meth. Engng. 11, 899–909.

Bischoff, M., Ramm, E., 1997. Shear deformable shell elements for large strains and deformations. Int. J. Num. Meth. Engng. 40, 4427–4449.

Boehler, J.P., 1987. Applications of tensor functions in solid mechanics. CISM courses and lectures, vol. 292. Springer.

Braun, M., 1995. Nichtlineare Analyse von geschichteten elastischen Flächentragwerken.

Ciarlet, P.G., 1988. Mathematical Elasticity, Three-dimensional Elasticity, vol. I. North-Holland, Amsterdam.

Eckstein, A., 1999. Finite-Simulation beliebiger 3D. Schalenstrukturen unter dem Einfluβ finiter inelastischer Deformationen und duktiler Schädigungsmechanismen bei Metallen. TWM Nr.99-3 Institut für konstruktiven Ingenieurbau, Ruhr-Universität Bochum.

Green, A.E., Adkins, J.E., 1960. Large Elastic Deformations and Non-linear Continuum Mechanics. Oxford University Press, Oxford.

Jones, E.M., 1975. Mechanics of Composite Materials. McGraw Hill, New York.

Ogden, R.W., 1972. Large deformation isotropic elasticity: on the correlation of theory and experiment for compressible rubber-like solids. Proc. Roy. Soc. London A 328, 567–683.

Schröder, J., 1996. Theoretische und algorithmische Konzepte zur phänomenologischen Beschreibung anisotropen Materialverhaltens.

Schultz, R., 1996. Nichtlineare Struktur- und Versagensanalyse von Faserverbundsystemen mit Mehrschichten-Schalenelementen. TWM Nr. 96-3. Institut für Konstruktiven Ingenieurbau, Ruhr-Universität.

Spencer, A.J.M., 1971. Theory and invariants. Eringen: Continuum Physics, vol. 1. Academic Press, New York.

Spencer, A.J.M., 1984. Constitutive theory for strongly ansiotropic solids. Continuum theory of the mechanics of fibre-reinforced composites. Springer, Berlin.

Wriggers, P., Reese, S., 1996. A note on enhanced strain methods for large deformations. Comp. Meth. Appl. Mech. Engng. 135, 201–209.

Xiao, H., 1996. On minimal representations for constitutive equations of anisotropic elastic materials. J. Elasticity 45, 13–32.